INFLUENCE OF VISCOSITY AND THERMAL CONDUCTIVITY ON PROPAGATION OF SOUND IMPULSES

(O VLIIANII VIAZKOSTI I TEPLOPROVODNOSTI NA RASPROSTRANENIE ZVUKOVYKH IMPUL'SOV)

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The first investigation of sound wave attenuation in a viscous gas is due to Stokes, the effect of thermal conductivity was taken into account later by Kirchhoff. A quite complete presentation of their results is available in the classic monograph of Lord Rayleigh [1]. From the very beginning Stokes, just as Kirchhoff, used the linear equations of acoustics as the starting point, and not the exact equations which describe the motion of continuous media.

In his studies of propagation of plane sound impulses in an ideal gas (without viscosity and thermal conductivity) Crussard showed [2] that asymptotic relationships of shock wave decay at large distances from the location of their origin are different from those for accoustic waves. Correct derivation of these relationships is not possible without consideration of nonlinear terms in equations of gas dynamics. Extension of Crussard's theory to cylindrical and spherical shock waves was given by Landau [3], Khristianovich [4], Sedov [5] and Whitham [6] using different methods.

It follows from the paper of Taylor [7] that the structure of weak shock waves is determined basically by convective processes, related to the nonlinear nature of Navier-Stokes equations, and by dissipation of energy at the expense of viscosity and thermal conductivity of real media. Therefore it appeared natural that both factors mentioned will influence the propagation of sound impulses to the same extent. This point of view was expressed by Lighthill [8]. He made a detailed analysis of this concept using plane motion as his example.

The decay of perturbations in cylindrical and spherical sound impulses is examined below. It turns out that the structure of waves and asymptotic relationships of their decay when time $t \to \infty$ are related to effects of viscosity and thermal conductivity. At this stage of the process, consideration of nonlinear terms in the Navier-Stokes equations may be neglected because their influence on the formation of the flow field is negligibly small. Variation of all gas parameters within the impulses occurs smoothly, shock waves are absent. Conversely, the motion of shock waves, as long as their width is much smaller than the general length of the wave, is determined by nonlinear convective terms of equations of gas dynamics.

When $t \to \infty$ the change of the maximum value of the excess pressure in N-waves, with consideration of viscosity and thermal conductivity, is inversely proportional to ~ for motions with axial symmetry and to t^2 for cenrally symmetric motions. The assertion of Lighthill [8] that asymptotic relationships of decay of perturbations must be exponential, turned out to be incorrect; the excess pressure varies according to an exponential law only in periodic sound waves with fixed wave length [1]. The conclusions obtained are based on a generalization of analysis of short waves carried out by Khristianovich [4] for unsteady one-dimensional motion of an ideal gas.

1. Equations of short waves. We shall examine one-dimensional flows for which all parameters depend on time t and on one single geometrical coordinate r, which determines the distance from the plane, the axis or the center of symmetry. Let v denote the velocity of particles, ρ the density, p the pressure, s the specific entropy, T the temperature, λ_1 the coefficient of viscosity, λ_2 the second coefficient of viscosity and k the coefficient of heat conduction. Continuity equations of Navier-Stokes and heat transfer equations are taken in the following form [9]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} + \frac{(\nu - 1)\rho v}{r} = 0 \tag{1.1}$$

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right) = -\frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left\{2\lambda_1\frac{\partial v}{\partial r} + \left(\lambda_2 - \frac{2}{3}\lambda_1\right)\left[\frac{\partial v}{\partial r} + \frac{(\nu-1)v}{r}\right]\right\} + \\ + \frac{2(\nu-1)\lambda_1}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \tag{1.2}$$

$$\rho T\left(\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial r}\right) = \frac{\partial}{\partial r}\left(k\frac{\partial T}{\partial r}\right) + \frac{(\nu-1)k}{r}\frac{\partial T}{\partial r} + 2\lambda_1\left[\left(\frac{\partial v}{\partial r}\right)^2 + (\nu-1)\left(\frac{v}{r}\right)^2\right] + \\ + \left(\lambda_2 - \frac{2}{3}\lambda_1\right)\left[\frac{\partial v}{\partial r} + \frac{(\nu-1)v}{r}\right] \tag{1.3}$$

Here $\nu = 0$, 1 and 2 for flows with plane, axial and central symmetry, respectively.

In order to close the system as written, two more equations which relate thermodynamic quantities ρ, p, s and T are added. As independent parameters we take density and pressure while specific entropy and temperature are expressed through these variables by means of the following differential relationships:

$$ds = \frac{c_p}{\alpha\rho a^2 T}(dp - a^2\,d\rho), \qquad dT = \frac{1}{\alpha\rho a^2}(\varkappa\,dp - a^2\,d\rho) \\ \left(\alpha = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p, \quad a^2 = \left(\frac{\partial p}{\partial\rho}\right)_s, \quad \varkappa = \frac{c_p}{c_v}\right) \tag{1.4}$$

which turned out to be quite convenient in the study of transonic flows of a viscous heat conducting gas [10]. In Equations (1.4) α designates the coefficient of thermal expansion, a the adiabatic sound velocity, $\varkappa$ the ratio of heat capacity at constant pressure c_p to heat capacity at constant volume c_v.

In the analysis of the system of Equations (1.1) to (1.4) the assumption is made that values of all gas parameters in the region of space under study differ slightly from the corresponding values in the equilibrium state. The latter will be designated by the index zero. A system of coordinates moving with the speed of sound a_0 in the unperturbed system is introduced and the characteristic length in this system is designated by L. We shall assume that the flow of gas under study represents a short wave, i.e. that the width of the region where the perturbations are concentrated is small comparison with the distances over which the wave propagates. This requirement is satisfied in most problems related to the investigation of explosive phenomena. With respect to perturbations of density, pressure, temperature and velocity of sound, we shall assume that they are of the same order of smallness as the mass velocity of particles. Changing to dimensionless variables we have

$$t = \frac{L}{\Delta a_0} t', \qquad r = a_0 t + Lr', \qquad v = \varepsilon a_0 v'$$
$$\rho = \rho_0 (1 + \varepsilon\rho'), \qquad p = p_0 (1 + \varepsilon p'), \qquad a = a_0 (1 + \varepsilon a') \tag{1.5}$$

Here ε and Δ are numerical parameters which in magnitude are considerably smaller than unity. As a result of substitution of relationships (1.5) into the system of equations (1.1) to (1.4), three dimensionless coefficients are obtained

$$N_{\mathrm{Re}1} = \frac{\rho_0 a_0 L}{\lambda_1}, \qquad N_{\mathrm{Re}2} = \frac{\rho_0 a_0 L}{\lambda_2}, \qquad N_{\mathrm{Pe}} = \frac{\rho_0 a_0 c_p L}{k}$$

In what follows we shall assume that the inverse quantities of these numbers have the same order of magnitude and are considerably smaller than one. In the derivation of approximate equations we shall retain in all relationships only the major terms neglecting other terms which have a higher order of smallness. Therefore, in equations of Navier-Stokes and in equations of heat transfer, coefficients of viscosity λ_1 and λ_2 and coefficient of heat conductivity k can be taken as constant and equal to their values in the equilibrium state of the medium. The introduction of small parameter Δ into determination of dimensionless time is connected with the assumption regarding the narrowness of the zone of perturbed motion. As a result of linearization of the equation of continuity we obtain (*)

$$\frac{\partial v}{\partial r} - \frac{\partial \rho}{\partial r} = 0$$

From equation of Navier-Stokes it follows that

$$\frac{\partial v}{\partial r} - \frac{p_0}{\rho_0 a_0^2} \frac{\partial p}{\partial r} = 0$$

Integration of last two equations leads to the following Formulas:

$$v = \rho = \frac{p_0}{\rho_0 a_0^2} p \tag{1.6}$$

which express the fact that in the approximation examined by us, the compression of gas occurs adiabatically and that the relationship of Riemann which characterizes a plane moving sound impulse [9], applies. It is known that the relationship of Riemann is also valid for weak shock waves [9].

The conclusion drawn appears as a direct consequence of not only the assumption of smallness in deviations of parameters of the medium in the field of perturbations as compared to corresponding values in the equilibrium state and the assumption about narrowness of the region of perturbed motion,

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*) Primes above dimensionless variables are omitted here and in what follows.

but also the assumption of Reynolds number values which are large in comparison with unity. In the simplification of the first two equations from the system (1.1) to (1.4) expressions were obtained in this manner which characterize the motion of ideal media. The influence of dissipative factors must be taken into account in the simplification of the heat transfer equation. Preliminary transformation of this equation is carried out to eliminate quantities of the first order of smallness which are related to mass flow and momentum of substance. Changing in Equation (1.3) from entropy and temperature to density and pressure by means of Equations (1.4) and combining the obtained expression with Equations (1.1) and (1.2) we have the required relationship

$$\frac{\partial p}{\partial t} + a_0 \frac{\partial p}{\partial r} + [(a_0 - v)^2 - a^2]\left(\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial r}\right) - \rho(a_0 - v)\left(\frac{\partial v}{\partial t} + a_0 \frac{\partial v}{\partial r}\right) + \\ + \frac{(\nu - 1)\rho v (a_0 - v)^2}{r} = (a_0 - v) L_1(\lambda_1, \lambda_2) + \frac{\alpha a^2}{c_p} L_2(k, \lambda_1, \lambda_2) \qquad (1.7)$$

Here the right-hand part of Equation (1.2) without the first term is denoted by $L_1(\lambda_1, \lambda_2)$ and right-hand part of Equation (1.3), by $L(k, \lambda_1, \lambda_2)$ After transition to a moving system of coordinates, partial derivatives of functions v and p with respect to the space coordinate disappear in the left-hand side of Equation (1.7).

In the approximation under consideration

$$da = \left(\frac{\partial a}{\partial \rho_0}\right)_s d\rho = \frac{(m_0 - 1) a_0}{\rho_0} d\rho \qquad \left(m_0 = \frac{1}{2\rho_0^3 a_0^2}\left(\frac{\partial^2 p}{\partial V_0^2}\right)_s, \quad V = \frac{1}{\rho}\right)$$

Using the last relationships, substituting Equations (1.5) into (1.7) and retaining only the higher terms in this equation, we obtain

$$m_0 \varepsilon v \frac{\partial v}{\partial r} + \Delta\left(\frac{\partial v}{\partial t} + \frac{\nu - 1}{2}\frac{v}{t}\right) = \frac{1}{2N_{\mathrm{Re}}}\left(1 + \frac{\varkappa - 1}{N_{\mathrm{Pr}}}\right)\frac{\partial^2 v}{\partial r^2} \qquad (1.8)$$

The total Reynolds number N_{Re} appearing in the last equation is connected with the so-called "longitudinal viscosity"

$$\frac{1}{N_{\mathrm{Re}}} = \frac{4}{3}\frac{1}{N_{\mathrm{Re}\,1}} + \frac{1}{N_{\mathrm{Re}\,2}}$$

The Prandtl number N_{Pr} is equal to the ratio of Péclet number N_{Pe} to Reynolds number N_{Re}. The order of Péclet and Reynolds numbers is the same according to assumption; therefore the Prandtl number will be of the order of unity. We note that terms in Equation (1.3) related to energy dissipation due to viscous forces do not influence the expression on the right-hand side of Equation (1.8).

Equation (1.8) governs the laws of motion of short waves in media in which dissipation of energy occurs. The analysis performed leads to generalization of results of Kristianovich [4] pertaining to unsteady one-dimensional gas flow without viscosity and thermal conductivity. We shall examine various specific cases in more detail.

2. Shock waves in ideal media. Let us assume at first that $\varepsilon \ll \Delta$ and $N_{\mathrm{Re}}^{-1} \ll \Delta$, then it follows from Equation (1.8) that

$$\frac{\partial v}{\partial t} + \frac{\nu-1}{2}\frac{v}{t} = 0$$

Integration of this relationship yields the well-known law of geometrical acoustics [9]

$$v = f(r)\, t^{-\frac{\nu-1}{2}} \tag{2.1}$$

which governs the propagation of plane, cylindrical and spherical waves. The function $f(r)$ in Equation (2.1) can be chosen arbitrarily. As is evident from the analysis presented, it is only possible to utilize results of geometrical acoustics so long as parameters of sound impulses vary sufficiently smoothly with time and space. However, Equation (2.1) does not give asymptotic relationships for damping of perturbations at $t \to \infty$ even for ideal media deprived of viscosity and thermal conductivity. It is easy to become convinced of this by substituting Equation (2.1) into original Equation (1.8).

Asymptotic relationships for damping of sonic impulses in ideal media are obtained under the condition that $N_{\mathrm{Re}}^{-1} \ll \varepsilon \sim \Delta$. Assuming for simplicity $m_0 \varepsilon = \Delta$, we have the following equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{\nu-1}{2}\frac{v}{t} = 0 \tag{2.2}$$

which is analogous to the one derived by Kristianovich [4] in different variables.

Performing the elementary substitution

$$\tau = \int t^{-\frac{\nu-1}{2}}\, dt, \qquad t^{\frac{\nu-1}{2}} v = u \tag{2.3}$$

we write Equation (2.2) in the form of plane wave equations

$$\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial r} = 0$$

Its integral containing an arbitrary function $g(u)$ is

$$\tau u - r = g(u) \tag{2.4}$$

Relationship (2.4) describes a simple Riemann [9] wave with linear characteristics. In order to find asymptotic relationships of damping of a sound impulse with a weak shock wave, it is sufficient to limit oneself to the special case $g(u) = 0$, since the distribution of all parameters of the gas with respect to a system of coordinates which moves together with the wave, at $t \to \infty$ will be given by a linear function of the geometrical coordinate [2 to 6]. Returning to previous variables according to Equations (2.3) we have

$$v = \frac{r}{t} \text{ for plane waves} \qquad v = \frac{r}{2t} \text{ for cylindrical waves} \qquad v = \frac{r}{t\ln t} \text{ for spherical waves} \tag{2.5}$$

We shall write expressions governing speed N of propagation of a weak shock wave in a quiescent gas. In dimensional variables [9]

$$N = a_0 + \frac{1}{2}\frac{m_0}{\rho_0 a_0}(p - p_0)$$

Taking into account Equations (1.5) and (1.6) we obtain $dr/dt = \frac{1}{2}v$

Differentiating relationships (2.5) along the trajectory of the shock wave front we find from here equations which determine the amplitude of the wave v_* at different instants of time. The solution of indicated equation leads to known results [2 to 6]

$$v_* = \frac{c}{t^{1/2}} \text{ for plane waves} \qquad v_* = \frac{c}{t^{3/4}} \text{ for cylindrical waves} \qquad v_* = \frac{c}{t\sqrt{\ln t}} \text{ for spherical waves} \qquad (2.6)$$

where c designates the constant of integration. Within the approximation examined, the excess pressure p_* at the shock front is proportional to the velocity of particles and therefore its change is subject to relationships (2.6).

Under the influence of dissipative factors shock waves will be gradually washed out at large distances from the point of origin. As long as the width of shock waves remains much smaller than the general length of the sound impulse, their motion is governed basically by terms in the left-hand part of Equation (1.8). Among these is the nonlinear term $v\partial v/\partial r$, which depends on taking into account of convective derivatives in Navier-Stokes equations. In the investigation of plane motion of gas, Lighthill found [8] that the variation in the maximum value of the velocity of particles follows the first of Equations (2.6) only when the sound impulse consists of one phase of compression. For this condition the relationship of width of the shock wave to the length of impulse is preserved as constant with respect to time. However, if the compression phase in the wave is followed by a rarefaction wave, then at $t \to \infty$ the maximum value of the velocity of particles approaches zero considerably faster than is predicted by the theory of shock wave propagation in ideal media [8]. The shock wave itself finally washes out completely and disappears.

3. Asymptotic relationships of damping of sound impulses. In order to obtain the asymptotic form which the sound impulses acquire for $t \to \infty$ under the influence of viscosity and thermal conductivity, let us examine the other limiting case $\varepsilon \ll \Delta \sim N_{\mathrm{Re}}^{-1}$. Let

$$\Delta = \frac{1}{2N_{\mathrm{Re}}}\left(1 + \frac{\varkappa - 1}{N_{\mathrm{Pr}}}\right)$$

We shall examine the behavior of N-waves in the case where the initial compression of gas is subsequently followed by an expansion; in problems with axial and central symmetry they are of fundamental interest [9]. However, the assumption made about the relative order of small quantities in relationships (1.5) and Equation (1.8), will be valid also for the plane motion of gas. We have

$$\frac{\partial v}{\partial t} + \frac{\nu - 1}{2}\frac{v}{t} = \frac{\partial^2 v}{\partial r^2}$$

Introducing a new sought function u according to the second of Equations (2.3), we obtain for determination of this function the classic equation of heat conduction

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2}$$

For description of N-waves it is necessary to take solutions of the dipole type [11]

$$n = \frac{h_1 r}{t^{3/2}} e^{-r^2/4t}$$

from which, changing back to dimensionless velocity of gas particles, we find

$$v = \frac{h_1 r}{t^{(\nu+2)/2}} e^{-r^2/4t} \tag{3.1}$$

Differentiating Equation (3.1) for t = const we see that the value of perturbed velocity will be a maximum when $r = \sqrt{2t}$. The minimum value of the velocity of particles is obtained for $r = -\sqrt{2t}$. Designating the maximum value of velocity which can be reached by particles in the wave, by an asterisk subscript, we have

$$v_* = \frac{h_2}{t} \text{ for plane waves} \qquad v_* = \frac{h_2}{t^{3/2}} \text{ for cylindrical waves} \qquad v_* = \frac{h_2}{t^2} \text{ for spherical waves} \tag{3.2}$$

where the constant h_2 is introduced instead of constant h_1 which is proportional to h_2. Comparison of Equations (2.6) and (3.2) shows that under the influence of viscosity and heat conduction the amplitude of sound impulses tends to zero significantly faster than expected from those relationships which are based on the assumption of the possibility to neglect these effects. Shock waves are absent in the flow. The excess pressure in the wave also varies in accordance with Equations (3.2). The length l is proportional to $\sqrt{t}$ not only for the plane, but also for the cylindrical and spherical sound impulses. In the propagation of impulses in ideal media $l \sim \sqrt{t}$ only for plane motion; in case of motion with axial symmetry $l \sim t^{\frac{1}{4}}$, and for spherically symmetric motion $l \sim \sqrt{\ln t}$, as we demonstrated in [2 to 6]. Thus, in a space with any number of dimensions the length of N-waves will vary according to one and the same relationship after the structure of flow begins to be defined primarily by dissipative factors.

Substitution of solution (3.2) into original equation (1.8) confirms that it is valid to neglect the term $v\partial v/\partial r$ in the left-hand part of the equation for $t \to \infty$ and for a no matter how small but different from zero value of reciprocal Reynolds number N_{Re}. In other words, the asymptotic form of sound impulses and their damping relationships are connected with viscosity and thermal conductivity effects. In the finite region of the wave propagation process, consideration of nonlinear terms in Navier-Stokes equations may be omitted.

We note that the first of Equations (3.2) follows from the paper of Lighthill [8]; however, the assertion made by Lighthill in the same paper that asymptotic damping relationship of perturbations must be exponential for cylindrical and spherical waves is incorrect. It is evident from Equations (3.2) that these are power laws. Velocity of particles and excess pressure vary according to exponential law only in periodic sound oscillations with fixed wave length [1].

Attention should be directed to one peculiarity inherent in the problem under examination. In spite of the fact that the structure of flow is fundamentally determined by dissipative factors and is described by the equation of heat conduction, the propagation of perturbations even in the final stage of the process takes place according to Equations (1.5) with adiabatic flow velocity a_0 in the quiescent medium. More exactly, the surface separating the compression phase form the expansion phase in the wave moves with such a velocity. Riemann relationships for isentropic simple waves in the first approximation remain valid in this surface.

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